The worksheet explores the numerical solution of Fluid flow, heat and mass transfer problems.

1. Fluid Flow

Obtain the relevant velocity field for each of the non-isothermal flows:

(a)

$$\mathrm{Re}\,\frac{\partial u}{\partial t} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2} + F_{body},$$

Subject to constant pressure (Couette flow) and the initial and boundary conditions:

$$\begin{aligned} &u(0,y) = 0, \quad 0 \leq y \leq 1; \\ &u(t,0) = 0, \quad u(t,1) = 1, \qquad t > 0. \end{aligned}$$

You are required to use the following generalized finite difference algorithm in which the cases $\alpha = 0,\ 1/2,\ 1$ *respectively corresponds to the explicit, semi-implicit (Crank-Nicolson) and implicit schemes:*

$$\mathrm{Re}\,\frac{u^{(n+1)} - u^{(n)}}{\Delta t} = u_{yy}^{(n+\alpha)} + F_{body}^{(n)},$$

Plot graphs that illustrate convergence to steady solutions.

(b)

$$\mathrm{Re}\,\frac{\partial w}{\partial t} = -\frac{dp}{dz} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial w}{\partial r}\right) + F_{body}, \qquad \frac{dp}{dz} = Const < 0.$$

Subject to the initial and boundary conditions:

$$\begin{aligned} &w(0,r) = 0, \quad 0 \leq r \leq 1; \\ &w(t,0) = 0, \quad \frac{\partial w}{\partial r}(t,1) = 0. \end{aligned}$$

You are required to use the following semi-implicit finite difference algorithm:

$$\mathrm{Re}\,\frac{w^{(n+1)} - w^{(n)}}{\Delta t} = -\frac{dp}{dz} + w_{rr}^{(n+\alpha)} + \frac{1}{r}\,w_r^{(n)} + F_{body}^{(n)}.$$

Plot graphs that illustrate convergence to steady solutions (use $dp/dz = -1$ and $\alpha = 1/2,\ 1$).

2. **Fluid flow and heat transfer**

$$
\begin{aligned}
\mathrm{Re}\,\frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\mu(T)\frac{\partial w}{\partial r}\right) + F_{body},\\
\mathrm{Re}\ \mathrm{Pr}\,\frac{\partial T}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) + \mu(T)\,\mathrm{Br}\left(\frac{\partial w}{\partial r}\right)^2 + \lambda\,H(T),
\end{aligned}
$$

$$
\mu(T) = \exp\left(-\frac{T}{1+\beta T}\right), \quad H(T) = \exp\left(\frac{T}{1+\beta T}\right).
$$

$$
\begin{aligned}
&w(0,r) = 0, \quad 0 \le r \le 1; \qquad \frac{\partial w}{\partial r}(t,0) = 0, \quad w(t,1) = 0,\\
&T(0,r) = 0, \quad 0 \le r \le 1; \qquad \frac{\partial T}{\partial r}(t,0) = 0, \quad \frac{\partial T}{\partial r}(t,1) = -\mathrm{Bi}(T(t,1) - T_a),
\end{aligned}
$$

Employ the semi-implicit finite difference algorithm:

$$
\mathrm{Re}\,\frac{w^{(n+1)} - w^{(n)}}{\Delta t} = G + \left[\frac{\partial \mu}{\partial r}\frac{\partial w}{\partial r} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right]^{(n)} + \mu^{(n)}\,\frac{\partial^2 w}{\partial r^2}^{(n+\alpha)} + F_{body}^{(n)},
$$

$$
\mathrm{Re}\ \mathrm{Pr}\,\frac{T^{(n+1)} - T^{(n)}}{\Delta t} = \frac{\partial^2 T}{\partial r^2}^{(n+\alpha)} + \frac{1}{r}\frac{\partial T}{\partial r}^{(n)} + \lambda H(T)^{(n)} + \mu^{(n)}\mathrm{Br}\left(\frac{\partial w}{\partial r}^{(n)}\right)^2.
$$

The above scheme simplifies as:

$$
-\Gamma_1\,w_{j-1}^{(n+1)} + (\,\mathrm{Re}\, + 2\Gamma_1)\,w_j^{(n+1)} - \Gamma_1\,w_{j+1}^{(n+1)} = \Gamma_2\,w_{j-1}^{(n)} + (\,\mathrm{Re}\, - 2\Gamma_2)\,w_j^{(n)} + \Gamma_2\,w_{j+1}^{(n)} + \Delta t\,G
$$

$$
+\frac{\Delta t}{2\Delta r}\left(w_{i+1}^{(n)} - w_{i-1}^{(n)}\right)\left[\frac{\left(\mu_{i+1}^{(n)} - \mu_{i-1}^{(n)}\right)}{2\Delta r} - \frac{\mu_i^{(n)}}{r_i}\right] + F_{body}^{(n)},
$$

$$
\begin{aligned}
-\Gamma_3\,T_{j-1}^{(n+1)} + (\,\mathrm{Re}\ \mathrm{Pr}\, + 2\Gamma_3)\,T_j^{(n+1)} - \Gamma_3\,T_{j+1}^{(n+1)} &= \Gamma_4\,T_{j-1}^{(n)} + (\,\mathrm{Re}\ \mathrm{Pr}\, - 2\Gamma_4)\,T_j^{(n)} + \Gamma_4\,T_{j+1}^{(n)}\\
&+\lambda\Delta t H(T)^{(n)} + \frac{\Delta t}{2r_i\Delta r}\left(T_{i+1}^{(n)} - T_{i-1}^{(n)}\right) + \frac{\mathrm{Br}\Delta t}{4\Delta r^2}\,\mu_i^{(n)}\left(u_{i+1}^{(n)} - u_{i-1}^{(n)}\right)^2,
\end{aligned}
$$

where

$\Gamma_1 = \alpha\,\mu^{(n)}\,\Delta t/(\Delta r)^2, \quad \Gamma_2 = (1-\alpha)\,\mu^{(n)}\,\Delta t/(\Delta r)^2, \quad \Gamma_3 = \alpha\,\Delta t/(\Delta r)^2, \quad \Gamma_4 = (1-\alpha)\,\Delta t/(\Delta r)^2.$